Extended quadratic algebra and a model of the equivariant cohomology ring of flag varieties

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Abstract

For the root system of type A we introduce and studied a certain extension of the quadratic algebra invented by S. Fomin and the first author, to construct a model for the equivariant cohomology ring of the corresponding flag variety. As an application of our construction we describe a generalization of the equivariant Pieri rule for double Schubert polynomials. For a general finite Coxeter system we construct an extension of the corresponding Nichols-Woronowicz algebra. In the case of finite crystallographic Coxeter systems we present a construction of extended Nichols-Woronowicz algebra model for the equivariant cohomology of the corresponding flag variety.

1 Introduction

In the paper [6] S. Fomin and the first author have introduced and study a model for the cohomology ring of flag varieties of type A as a commutative subalgebra generated by the so-called Dunkl elements in a certain noncommutative quadratic algebra $\mathcal{E}_n$. An advantage of the approach developed in [6] is that it admits a simple generalization which is suitable for description of the quantun cohomology ring of flag varieties, as well as (quantum) Schubert polynomials. Constructions from the paper [6] have been generalized to other finite root systems by the authors in [9]. One of the main constituents of the above constructions is the Dunkl elements. The basic properties of the Dunkl elements are:

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1) they are pairwise commuting;

2) in the so-called Calogero-Moser representation [6, 9] they correspond to the *truncated* (i.e. without differential part) rational Dunkl operators [4];

3) in the crystallographic case they correspond – after applying the so-called Bruhat representation [6, 9]– to the Monk formula in the cohomology ring of the flag variety in question;

4) in the crystallographic case, subtraction-free expressions of Schubert polynomials calculated at the Dunkl elements in the algebra $\widetilde{\mathcal{BE}}(\Delta)$ should provide a combinatorial rule for describing the Schubert basis structural constants, i.e. the intersection numbers of Schubert classes.

In the case of classical root systems Δ, the first author [7] has defined a certain extension $\widetilde{\mathcal{BE}}(\Delta)$ of the algebra $\mathcal{BE}(\Delta)$ together with a pairwise commuting family of elements, called Dunkl elements, which after applying the Calogero-Moser representation exactly coincide with the rational Dunkl operators. One of the main objective of our paper is to study a commutative subalgebra generated by the Dunkl elements in the extended algebra $\widetilde{\mathcal{BE}}(\Delta)$ in the case of type A root systems. Postnikov [13] proved a Pieri-type formula for the elementary symmetric polynomial evaluated at the Dunkl elements, which was originally conjectured in [6]. In Theorem 2.1 we show an analogue of the Pieri-type formula in the extended algebra $\widetilde{\mathcal{BE}}(\Delta)$ of type A. As a consequence, it is shown that the commutative subalgebra generated by the Dunkl elements is isomorphic to the equivariant cohomology ring $H_T^*(Fl_n)$, where Fl_n is the flag variety parametrizing the full flags in the vector space $\mathbf{C}^n$ and $T = (\mathbf{C}^\times)^n$ is the torus acting on $\mathbf{C}^n$ diagonally.

In Section 3 we construct the Bruhat representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ and study some properties of the former. The existence of Bruhat's representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ plays a crucial role in applications to the equivariant Schubert calculus, and constitutes an important step in the construction of the model of $H_T^*(Fl_n)$. Our formula in Theorem 2.1 describes the Pieri formula for the equivariant cohomology ring $H_T^*(Fl_n)$ via the Bruhat representation. As shown in Section 4, our formula also covers the quantized case. The Pieri formula for double Schubert polynomials was studied in [15, Chapter 5]. A similar rule in the cohomology ring $H_T^*(Fl_n)$ was stated and proved in [14]. We will show how to compute the multiplication by the special Schubert classes via the Bruhat representation in Example 3.1, where an example is also given to see that the result of the computation based on our formula coincides with the one from [14].

Another objective of our paper is to construct a certain extension of the Nichols-Woronowicz model for the coinvariant algebra of a finite Coxeter group W. It is conjectured that the Fomin-Kirllov quadratic algebra $\mathcal{E}_n$ is isomorphic to the Nichols-Woronowicz algebra associated to a certain kind of Yetter-Drinfeld module defined by the data of the root system of type A_{n-1}. More generally, the algebra $\mathcal{BE}(\Delta)$ is a lift of the Nichols-Woronowicz algebra for the root system Δ. Recall that the Nichols-Woronowicz algebra model for the cohomology ring of flag varieties has been invented by Y. Bazlov [2]. In Section 4 we introduce a certain extension $\widetilde{\mathcal{B}}_W$ of the Nichols-Woronowicz algebra $\mathcal{B}_W$ and construct a commutative subalgebra in the extended Nichols-Woronowicz algebra. Our second main result states that, for crystallographic root systems and $t=0$, the commutative subalgebra of $\widetilde{\mathcal{B}}_W$ in question is isomorphic to the T-equivariant cohomology ring of the corresponding flag variety.

2 Extension of the quadratic algebra

Definition 2.1 *The algebra $\mathcal{E}_n$ is an associative algebra generated by the symbols $[i,j]$, $1 \leq i,j \leq n$, $i \neq j$, subject to the relations:*
(0) : $[i,j] = -[j,i]$
(1) : $[i,j]^2 = 0$,
(2) : $[i,j][k,l] = [k,l][i,j]$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$,
(3) : $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

Let us consider the extension $\mathcal{E}_n\langle R\rangle[t]$ of the quadratic algebra $\mathcal{E}_n$ by the polynomial ring $R[t] = \mathbf{Z}[x_1,\ldots,x_n][t]$ defined by the commutation relations:
(A): $[i,j]x_k = x_k[i,j]$, for $k \neq i,j$,
(B): $[i,j]x_i = x_j[i,j] + t$, $[i,j]x_j = x_i[i,j] - t$, for $i<j$,
(C): $[i,j]t = t[i,j]$.

Note that the $\mathbf{S}_n$-invariant subalgebra $R^{\mathbf{S}_n}[t]$ of $R[t]$ is contained in the center of the algebra $\mathcal{E}_n\langle R\rangle[t]$. Let $e_k(x_1,\ldots,x_n)$, $1 \leq k \leq n$, stands for the elementary symmetric polynomial of degree k in the variables $x_1,\ldots,x_n$. We put by definition, $e_0(x_1,\ldots,x_n) = 1$, and $e_k(x_1,\ldots,x_n) = 0$, if $k<0$.

Definition 2.2 (1) *We define the $R[t]$-algebra $\widetilde{\mathcal{E}}_n[t]$ by*

$$\widetilde{\mathcal{E}}_n[t] = \mathcal{E}_n\langle R\rangle[t] \otimes_{R^{\mathbf{S}_n}} R.$$

More explicitly, $\widetilde{\mathcal{E}}_n[t]$ *is an algebra over the polynomial ring* $\mathbf{Z}[y_1,\ldots,y_n]$ *generated by the symbols* $[i,j]$, $1 \le i,j \le n$, $i \neq j$, *and* $x_1,\ldots,x_n,t$ *satisfying the relations in the definition of the algebra* $\mathcal{E}_n\langle R\rangle[t]$, *together with the identification* $e_i(x_1,\ldots,x_n) = e_i(y_1,\ldots,y_n)$, *for* $i=1,\ldots,n$. *Denote by* $\widetilde{\mathcal{E}}_{n,t_0}$ *the specialization of* $\widetilde{\mathcal{E}}_n[t]$ *at* $t=t_0$.
(2) *The Dunkl elements* $\theta_i \in \widetilde{\mathcal{E}}_n[t]$, $i=1,\ldots,n$, *are defined by the formula*

$$\theta_i = \theta_i^{(n)} = x_i + \sum_{j\neq i}[i,j].$$

Remark 2.1 Note that x_i's do not commute with the Dunkl elements, but only symmetric polynomials in x_i's do. Moreover, we need the R-algebra structure of $\widetilde{\mathcal{E}}_n[t]$ to construct the model of the T-equivariant cohomology ring $H_T^*(Fl_n)$ which is an algebra over $H_T^*(\mathrm{pt.}) \cong R$. By this reason we need the second copy of $R = \mathbf{Z}[y_1,\ldots,y_n]$, where y_i's are assumed to belong to the center of the algebra $\widetilde{\mathcal{E}}_n[t]$, and $f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$ for any symmetric polynomial f.

Lemma 2.1 *The Dunkl elements commute each other.*

Proof. This follows from the fact that

$$(x_i+x_j)[i,j] = [i,j](x_i+x_j).$$

Theorem 2.1 (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$) *For* $k \le m \le n$, *we have*

$$e_k(\theta_1^{(n)},\ldots,\theta_m^{(n)}) =$$
$$\sum_{r\ge 0}(-t)^r\ N(m-k,2r)\Big\{\sum_{S,I=\{i_a\},(j_a)} X_S\cdot[i_1,j_1]\cdots[i_{|I|},j_{|I|}]\Big\},$$

where

$$N(a,2b) = (2b-1)!!\binom{a+2b}{2b},$$

$X_S := \prod_{s\in S} x_s$, *and the second summation runs over triples* $(S, I=\{i_1,\ldots,i_{|I|}\},(j_a)_{a=1}^{|I|})$ *such that* $S \subset \{1,\ldots,m\}$; I *is a subset of* $\{1,\ldots,m\}\setminus S$; $|I|+|S|+2r=k$; $1 \le i_a \le m < j_a \le n$; $j_1 \le \cdots \le j_{|I|}$.

Proof. Let $\mathcal{A}$ be a subset of $\{1,\ldots,n\}$, $m := |\mathcal{A}|$, $d := n-m$ and $\{1,\ldots,n\}\setminus\mathcal{A} = \{j_1 < \cdots < j_d\}$. Denote by $E_k(\mathcal{A})$ the right-hand side of the formula, i.e.,

$$E_k(\mathcal{A}) := \sum_{r\geq 0}(-t)^r N(m-k,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{(*)}[s_1,t_1]\cdots[s_{k-2r-|S|},t_{k-2r-|S|}],$$

where $(*)$ stands for the conditions that $s_1,\ldots,s_{k-2r-|S|}\in\mathcal{A}\setminus S$ are distinct, $t_1,\ldots,t_{k-2r-|S|}\in\{1,\ldots,n\}\setminus\mathcal{A}$ and $t_1\leq\cdots\leq t_{k-2r-|S|}$. It will suffice to prove the recursive formula

$$E_k(\mathcal{A}\cup\{j=j_1\}) = E_k(\mathcal{A}) + E_{k-1}(\mathcal{A})(x_j + \sum_{s\neq j}[j,s]).$$

For a subset $I=\{i_1,\ldots,i_l\}\subset\{1,\ldots,n\}$ and $p\notin I$, we use the symbol

$$\langle\!\langle I|p\rangle\!\rangle = \sum_{w\in\mathbf{S}_l}[i_{w(1)},p]\cdots[i_{w(l)},p]$$

as defined in [13]. We also use the symbol $I_1\cdots I_d\subset_m I$ which means that $I_1,\ldots,I_d\subset I$ are disjoint and $\#I_1+\cdots+\#I_d=m$. We have the following decompositions:

$$\begin{aligned}
&E_k(\mathcal{A})\\
&\quad= \sum_{r\geq 0}(-t)^r N(m-k,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{I_1\cdots I_d\subset_{k-2r-|S|}\mathcal{A}\setminus S}\langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
&\quad= \sum_{r\geq 0}(-t)^r N(m-k,2r)(A_1^r + A_2^r),\\
&E_k(\mathcal{A}\cup\{j\})\\
&\quad= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{I_2\cdots I_d\subset_{k-2r-|S|}\mathcal{A}\cup\{j\}\setminus S}\langle\!\langle I_2|j_2\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
&\quad= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(B_1^r + B_2^r + B_3^r),\\
&E_{k-1}(\mathcal{A})\sum_{s\neq j}[j,s]\\
&\quad= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S}\langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\sum_{s\neq j}[j,s]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(C_1^r+C_2^r+C_3^r+C_4^r),\\
&E_{k-1}(\mathcal{A})x_j\\
&= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S} \langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle x_j\\
&= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(D_1^r+D_2^r),
\end{aligned}
$$

where A_i^r, B_i^r, C_i^r, D_i^r are defined as follows.

- A_1^r is the sum of terms with $I_1=\emptyset$; A_2^r is the sum of terms with $I_1\neq\emptyset$.
- B_1^r is the sum of terms with $j\notin S\cup I_2\cup\cdots\cup I_d$; B_2^r is the sum of terms with $j\in I_2\cup\cdots\cup I_d$; B_3^r is the sum of terms with $j\in S$.
- C_1^r is the sum of terms with $s\in\mathcal{A}\setminus(S\cup I_1\cup\cdots\cup I_d)$; C_2^r is the sum of terms with $s\in I_2\cup\cdots\cup I_d\cup\mathcal{A}^c$; C_3^r is the sum of terms with $s\in S$; C_4^r is the sum of terms with $s\in I_1$.
- D_1^r is the sum of terms with $I_1=\emptyset$; D_2^r is the sum of terms with $I_1\neq\emptyset$.

Based on the same arguments used in [13], we can see that $A_1^r=B_1^r$, $A_2^r+C_1^r=0$, $B_2^r=C_2^r$ and $C_4^r=0$. It is also easy to see that $B_3^r=D_1^r$. Now we have

$$
\begin{aligned}
&E_k(\mathcal{A})+E_{k-1}(\mathcal{A})(x_j+\sum_{s\neq j}[j,s])-E_k(\mathcal{A}\cup\{j\})\\
&= \sum_{r\geq 0}(-t)^r\left(N(m-k,2r)(A_1^r+A_2^r)-N(m-k+1,2r)(B_1^r-C_1^r-C_3^r-D_2^r)\right)\\
&= \sum_{r\geq 1}(-t)^r\left(N(m-k,2r)-N(m-k+1,2r)\right)(A_1^r+A_2^r)\\
&\quad+\sum_{r\geq 0}(-t)^r N(m-k+1,2r)(C_3^r+D_2^r).
\end{aligned}
$$

From the commutation relation $[i,j]x_j=x_i[i,j]-t$, we have

$$
D_2^r = \sum_{S\subset\mathcal{A}} X_S \sum_{\substack{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S\\ I_1=\{a_1,\ldots,a_{|I_1|}\}}} \sum_{w\in\mathbf{S}_{|I_1|}} x_{a_{w(|I_1|)}}[a_{w(1)},j]\cdots[a_{w(|I_1|)},j]\langle\!\langle I_2|j_2\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle
$$

$$
\begin{aligned}
& -t\sum_{S\subset\mathcal{A}} X_S \sum_{\substack{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S\\ I_1=\{a_1,\ldots,a_{|I_1|}\}}} \sum_{w\in\mathbf{S}_{|I_1|}} [a_{w(1)},j]\cdots[a_{w(|I_1|-1)},j]\langle\!\langle I_2|j_2\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
= & \sum_{S\subset\mathcal{A}}\sum_{s\notin S} X_{S\cup\{s\}} \sum_{I_1\cdots I_d\subset_{k-1-2r-(|S|+1)}\mathcal{A}\setminus S\cup\{s\}} \langle\!\langle I_1|j_1\rangle\!\rangle[s,j]\langle\!\langle I_2|j_2\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
& -(m-k+2r+2)t\sum_{S\subset\mathcal{A}} X_S \sum_{I_1\cdots I_d\subset_{k-2-2r-|S|}\mathcal{A}\setminus S} \langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
= & -C_3^r+(-t)(m-k+2r+2)(A_1^{r+1}+A_2^{r+1}).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& (-t)^{r+1}\left(N(m-k,2(r+1))-N(m-k+1,2(r+1))\right)(A_1^{r+1}+A_2^{r+1})\\
&= -(-t)^{r+1}(2r+1)!!\frac{(m-k+2r+2)!}{(2r+1)!(m-k+1)!}(A_1^{r+1}+A_2^{r+1})\\
&= -(-t)^{r}(2r-1)!!\frac{(m-k+2r+1)!}{(2r)!(m-k+1)!}\cdot(-t)(m-k+2r+2)(A_1^{r+1}+A_2^{r+1})\\
&= -(-t)^{r}N(m-k+1,2r)(C_3^r+D_2^r).
\end{aligned}
$$

This shows the desired result.

Example 2.1 Let us check the coefficients of t and t^2 in the expression of $\theta_1\theta_2\theta_3\theta_4\in\mathcal{E}_5\langle R\rangle[t]$ by direct computation. It is easy to see

$$
\begin{aligned}
\theta_1\theta_2\theta_3 = & -t(x_1+x_2+x_3+[14]+[15]+[24]+[25]+[34]+[35])\\
& +x_1x_2x_3+x_1x_2([34]+[35])+x_1x_3([24]+[25])+x_2x_3([14]+[15])\\
& +x_1([24][34]+[24][35]+[25][35]+[34][24]+[34][25]+[35][25])\\
& +x_2([14][34]+[14][35]+[15][35]+[34][14]+[34][15]+[35][15])\\
& +x_3([14][24]+[14][25]+[15][25]+[24][14]+[24][15]+[25][15])\\
& +\sum_{\{i_1,i_2,i_3\}=\{1,2,3\},\ 4\le j_1\le j_2\le j_3\le 5} [i_1j_1][i_2j_2][i_3j_3].
\end{aligned}
$$

Multiply this expression by θ_4 from the right. We have the contributions to the coefficients of t and t^2 only from the following terms:

- $$
\begin{aligned}
& -t\Big(x_1+x_2+x_3+[14]+[15]+[24]+[25]+[34]+[35]\Big)\theta_4\\
&= 3t^2-t\Big(x_1x_4+x_2x_4+x_3x_4+x_1([42]+[43]+[45])
\end{aligned}
$$

$$
\begin{aligned}
&+x_2([41]+[43]+[45])+x_3([41]+[42]+[45])+x_4([15]+[25]+[35])\\
&+[15][41]+[24][41]+[25][41]+[34][41]+[35][41]\\
&+[14][42]+[15][42]+[25][42]+[34][42]+[35][42]\\
&+[14][43]+[15][43]+[24][43]+[25][43]+[35][43]\\
&+[14][45]+[15][45]+[24][45]+[25][45]+[34][45]+[35][45]\Big),
\end{aligned}
$$

- $x_1([24][34]+[24][35]+[25][35]+[34][24]+[34][25]+[35][25])x_4$
  $= \; -tx_1([24]+[34]+[25]+[35])+\cdots,$
- $x_2([14][34]+[14][35]+[15][35]+[34][14]+[34][15]+[35][15])x_4$
  $= \; -tx_2([14]+[34]+[15]+[35])+\cdots,$
- $x_3([24][14]+[24][15]+[25][15]+[14][24]+[14][25]+[15][25])x_4$
  $= \; -tx_3([24]+[14]+[25]+[15])+\cdots,$

- $x_1x_2([34]+[35])x_4 = -tx_1x_2+x_1x_2(x_3[34]+x_4[35]),$
- $x_2x_3([14]+[15])x_4 = -tx_2x_3+x_2x_3(x_1[14]+x_4[15]),$
- $x_1x_3([24]+[25])x_4 = -tx_1x_3+x_1x_3(x_2[24]+x_4[25]),$

- $$
\begin{aligned}
&\sum_{\{i_1,i_2,i_3\}=\{1,2,3\},\ 4\le j_1\le j_2\le j_3\le 5}[i_1j_1][i_2j_2][i_3j_3]x_4\\
= \; &-t([14][24]+[14][35]+[25][35]+[24][14]+[24][35]+[15][35]\\
&+[34][14]+[34][25]+[15][25]+[34][24]+[34][15]+[25][15]\\
&+[14][34]+[14][25]+[35][25]+[24][34]+[24][15]+[35][15])+\cdots.
\end{aligned}
$$

By using the relations $[15][41]+[14][45]=[45][15]$, $[25][42]+[24][45]=[45][25]$, $[35][43]+[34][45]=[45][35]$, we obtain finally that

$$
\begin{aligned}
\theta_1\theta_2\theta_3\theta_4 \; = \; & 3t^2-t\Big(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4\\
&+x_1([25]+[35]+[45])+x_2([15]+[35]+[45])\\
&+x_3([15]+[25]+[45])+x_4([15]+[25]+[35])\\
&+[15][25]+[15][35]+[15][45]+[25][35]+[25][45]+[35][45]\\
&+[25][15]+[35][15]+[45][15]+[35][25]+[45][25]+[45][35]\Big)+\cdots.
\end{aligned}
$$

It is easy to see that the formula for $\theta_1\theta_2\theta_3\theta_4$ stated in Theorem 2.1 produces the same expression.

The following is a special case of the formula in Theorem 2.1 for $m = n$.

Corollary 2.1 *We have the relations*

$$e_k(\theta_1^{(n)}, \ldots, \theta_n^{(n)}) =$$

$$e_k(y_1, \ldots, y_n) + \sum_{r \geq 1} (-t)^r \ (2r-1)!! \binom{n-k+2r}{2r} e_{k-2r}(y_1, \ldots, y_n), \ 1 \leq k \leq n,$$

in the algebra $\widetilde{\mathcal{E}}_n[t]$.

It will be shown in Section 3 that the above relations describe the complete set of relations among the Dunkl elements in $\widetilde{\mathcal{E}}_n[t]$.

3 Bruhat representation

Let us recall the definition of the Bruhat representation of the algebra $\mathcal{E}_n$ on the group ring of the symmetric group $\mathbf{Z}\langle \mathbf{S}_n \rangle = \oplus_{w \in \mathbf{S}_n} \mathbf{Z} \cdot \underline{w}$. The operator σ_{ij}, $i < j$, is defined as follows:

$$\sigma_{ij}(\underline{w}) = \begin{cases} \underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $t_{ij} \in \mathbf{S}_n$ is the transposition of i and j. Then the Bruhat representation of $\mathcal{E}_n$ is defined by $[i, j].\underline{w} := \sigma_{ij}(\underline{w})$.

Now we extend the Bruhat representation to that of the algebra $\mathcal{E}_n\langle R \rangle[t]$ defined on

$$R[t]\langle \mathbf{S}_n \rangle = \oplus_{w \in \mathbf{S}_n} \mathbf{Z}[y_1, \ldots, y_n][t] \cdot \underline{w}.$$

Let us define the devided difference operator ∂_{ij} on $\mathbf{Z}[y_1, \ldots, y_n][t]$ as a $\mathbf{Z}[t]$-linear operator given by $\partial_{ij} := (1 - t_{ij})/(y_i - y_j)$. For $f(y) \in \mathbf{Z}[y_1, \ldots, y_n][t]$ and $w \in \mathbf{S}_n$, we define the $\mathbf{Z}[t]$-linear operators $\tilde{\sigma}_{ij}$, $i < j$, and ξ_k as follows:

$$\tilde{\sigma}_{ij}(f(y)\underline{w}) = \begin{cases} t(\partial_{w(i)w(j)} f(y))\underline{w} + f(y)\underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ t(\partial_{w(i)w(j)} f(y))\underline{w}, & \text{otherwise,} \end{cases}$$

$$\xi_k(f(y)\underline{w}) = (y_{w(k)} f(y))\underline{w}.$$

Proposition 3.1 *The algebra $\mathcal{E}_n\langle R\rangle[t]$ acts $\mathbf{Z}[t]$-linearly on $\mathbf{Z}[y][t]\langle \mathbf{S}_n\rangle$ via $[ij] \mapsto \tilde{\sigma}_{ij}$ and $x_k \mapsto \xi_k$.*

Proof. Let us check the compatibility with the defining relations of the algebra $\widetilde{\mathcal{E}_n}[t]$. We show the compatibility only with the relations (1), (3) and (B). The rest are easy to check.

Let us start with the relation (1). We have

$$\begin{aligned}\tilde{\sigma}_{ij}^2(f(y)\underline{w}) &= \tilde{\sigma}_{ij}\left(t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\sigma_{ij}(\underline{w})\right)\\ &= t^2(\partial^2_{w(i)w(j)}f(y))\underline{w} + t(\partial_{w(i)w(j)}f(y))\sigma_{ij}(\underline{w})\\ &\quad +t(\partial_{w(j)w(i)}f(y))\sigma_{ij}(\underline{w}) + f(y)\sigma_{ij}^2(\underline{w}).\end{aligned}$$

Since $\partial^2_{w(i)w(j)} = 0$, $\sigma_{ij}^2 = 0$ and $\partial_{w(i)w(j)} = -\partial_{w(j)w(i)}$, we get $\tilde{\sigma}_{ij}^2 = 0$.

For the relation (3), we have

$$\begin{aligned}\tilde{\sigma}_{ij}\tilde{\sigma}_{jk}(f(y)\underline{w}) &= \tilde{\sigma}_{ij}\left(t(\partial_{w(j)w(k)}f(y))\underline{w} + f(y)\sigma_{jk}(\underline{w})\right)\\ &= t^2(\partial_{w(i)w(j)}\partial_{w(j)w(k)}f(y))\underline{w} + t(\partial_{w(j)w(k)}f(y))\sigma_{ij}(\underline{w})\\ &\quad +t(\partial_{w(i)w(k)}f(y))\sigma_{jk}(\underline{w}) + f(y)\sigma_{ij}\sigma_{jk}(\underline{w}).\end{aligned}$$

We also obtain $\tilde{\sigma}_{jk}\tilde{\sigma}_{ki}(f(y)\underline{w})$ and $\tilde{\sigma}_{ki}\tilde{\sigma}_{ij}(f(y)\underline{w})$ by the cyclic permutation of i, j, k. The 3-term relations

$$\partial_{w(i)w(j)}\partial_{w(j)w(k)} + \partial_{w(j)w(k)}\partial_{w(k)w(i)} + \partial_{w(k)w(i)}\partial_{w(i)w(j)} = 0$$

and

$$\sigma_{ij}\sigma_{jk} + \sigma_{jk}\sigma_{ki} + \sigma_{ki}\sigma_{ij} = 0$$

show the desired equality

$$\tilde{\sigma}_{ij}\tilde{\sigma}_{jk} + \tilde{\sigma}_{jk}\tilde{\sigma}_{ki} + \tilde{\sigma}_{ki}\tilde{\sigma}_{ij} = 0.$$

Finally, we check the relation (B). We have

$$\begin{aligned}\tilde{\sigma}_{ij}\xi_i(f(y)\underline{w}) &= \tilde{\sigma}_{ij}(y_{w(i)}f(y)\underline{w})\\ &= t\partial_{w(i)w(j)}(y_{w(i)}f(y))\underline{w} + (y_{w(i)}f(y))\sigma_{ij}(\underline{w})\\ &= t(f(y)\underline{w}) + t(y_{w(j)}\partial_{w(i)w(j)}f(y))\underline{w} + y_{wt_{ij}(j)}\sigma_{ij}(\underline{w})\\ &= \xi_j\tilde{\sigma}_{ij}(f(y)\underline{w}) + t(f(y)\underline{w}).\end{aligned}$$

Now the proof of the well-definedness of the Bruhat representation is completed.

Let us denote by AH_n^0 the subalgebra in $\mathcal{E}_n\langle R\rangle[t]$ generated by the elements t, $h_1 := [1,2], h_2 := [2,3], \ldots, h_{n-1} := [n-1,n]$ and $x_1, \ldots, x_n$.

Lemma 3.1 *The module $R[t]\langle \mathbf{S}_n\rangle$ is generated by $\underline{\mathrm{id.}}$ over AH_n^0.*

Proof. If we take a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$, $s_i = (i, i+1)$, of a permutation $w \in \mathbf{S}_n$, then we have

$$h_{i_l}\cdots h_{i_1}(\underline{\mathrm{id.}}) = \underline{w}.$$

This shows $R[t]\langle \mathbf{S}_n\rangle = AH_n^0 \cdot \underline{\mathrm{id.}}$

Theorem 3.1 *The subalgebra AH_n^0 of $\mathcal{E}_n\langle R\rangle[t]$ is isomorphic to the nil degenerate affine Hecke algebra $\mathcal{AH}_n^0$ of type $A_{n-1}^{(1)}$, i.e. the $\mathbf{Z}[t]$-algebra given by two sets of generators $g_1, \ldots, g_{n-1}$ and $x_1, \ldots, x_n$ subject to the set of defining relations:*

$$g_i^2 = 0, \quad g_i g_j = g_j g_i, \ \text{ if } |i-j| > 1, \quad g_i g_j g_i = g_j g_i g_j, \ \text{ if } |i-j| = 1,$$
$$x_i x_j = x_j x_i, \quad x_k g_i = g_i x_k, \ \text{ if } k \neq i, i+1, \ \ g_i x_i - x_{i+1} g_i = t.$$

Proof. First of all it is easy to see that the elements $t, h_1, \ldots, h_{n-1}, x_1, \ldots, x_n$ do satisfy the relations listed above. Hence we have a surjective homomorphism ρ from the nil degenerate affine Hecke algebra $\mathcal{AH}_n^0$ to AH_n^0 given by $\rho(g_i) = h_i$. Now we are going to construct a basis in the algebra $\mathcal{AH}_n^0$. Let $w \in \mathbf{S}_n$ be a permutation and $w = s_{i_1}\cdots s_{i_k}$ its any reduced decomposition. Since the elements $g_1, \ldots, g_{n-1}$ satisfy the Coxeter relations, the element $g_w := g_{i_1}\cdots g_{i_k}$ is well-defined. On the other hand, using relations among the elements $x_1, \ldots, x_n$ and $g_1, \ldots, g_{n-1}$, in the algebra $\mathcal{AH}_n^0$, one can write any element of $\mathcal{AH}_n^0$ as a linear combination of elements $t^a x^m g_w$, where $w \in \mathbf{S}_n$, $m \in \mathbf{Z}_{\geq 0}^n$ and $a \in \mathbf{Z}_{\geq 0}$. Lemma 3.1 implies that $R[t]\langle \mathbf{S}_n\rangle = \rho(\mathcal{AH}_n^0)\cdot \underline{\mathrm{id.}}$, and hence the elements $\{t^a x^m g_w \mid a \in \mathbf{Z}_{\geq 0},\ w \in \mathbf{S}_n,\ m \in \mathbf{Z}_{\geq 0}^n\}$ must be linearly independent over $\mathbf{Z}$. This means the injectivity of ρ, so we conclude that the homomorphism ρ is isomorphism.

Let us consider the double Schubert polynomials $\mathfrak{S}_w(x,y)$, $w \in \mathbf{S}_n$, introduced by Lascoux and Schützenberger [10]. The polynomial $\mathfrak{S}_{w_0}(x,y)$ for the maximal element $w_0 \in \mathbf{S}_n$ is by definition given by $\mathfrak{S}_{w_0}(x,y) := \prod_{i+j\leq n}(x_i - y_j)$. For $w \in \mathbf{S}_n$, take a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$. The divided difference operator $\partial_w := \partial_{i_1}\cdots\partial_{i_l}$ is well-defined thanks to the Coxeter relations. We define the double Schubert polynomial for w by $\mathfrak{S}_w(x,y) := \partial^{(x)}_{w^{-1}w_0}\mathfrak{S}_{w_0}(x,y)$, where $\partial^{(x)}_w$ means a divided difference operator on x-variables. The Schubert polynomials are defined to be the specializations $\mathfrak{S}_w(x) := \mathfrak{S}_w(x,0)$ of the double Schubert polynomials.

Theorem 3.2 *Let $\mathfrak{S}_w(x,y)$ be the double Schubert polynomial corresponding to $w \in \mathbf{S}_n$. When $t=0$, we have*

$$\mathfrak{S}_w(\theta, y)(\underline{\mathrm{id.}}) = \underline{w}.$$

Proof. This follows from the Monk formula for the double Schubert polynomials, see e.g. [11, Exercise 2.7.2], and

$$\begin{aligned}(\theta_i - y_{w(i)})(\underline{w}) &= \xi_i(\underline{w}) + \sum_{j \neq i} \sigma_{ij}(\underline{w}) - y_{w(i)}\underline{w} \\ &= \sum_{j>i, l(wt_{ij})=l(w)+1} \underline{wt_{ij}} - \sum_{j<i, l(wt_{ij})=l(w)+1} \underline{wt_{ij}}.\end{aligned}$$

Let $w \in \mathbf{S}_n$, r be the maximal descent of w, and s be the greatest integer such that $w(s) < w(r)$. By applying the Monk formula to the product $x_r \mathfrak{S}_u(x,y)$, $u = wt_{rs}$, we get the transition formula

$$\mathfrak{S}_w(x,y) = (x_r - y_{u(r)})\mathfrak{S}_u(x,y) + \sum_{v \in S(w,r)} \mathfrak{S}_v(x,y),$$

where $S(w,r)$ is the set of permutations of form $ut_{jr} = wt_{rs}t_{jr}$ with $j < r$ and of the same length as w. Similarly we have

$$\underline{w} = (\theta_r - y_{u(r)})\underline{u} + \sum_{v \in S(w,r)} \underline{v}.$$

Let r' be the maximal descent of $v = wt_{rs}t_{jr} \in S(w,r)$. Since $l(v) = l(wt_{rs}) + 1$, we have $v(r) = wt_{rs}(j) < wt_{rs}(r) = w(s) < w(r)$. Hence we have $r' < r$ or "$r' = r$ and $v(r) < w(r)$". Then our assertion is proved by induction on the Bruhat ordering, the maximal descent r and $w(r)$.

Remark 3.1 Only when $t=0$, one can extend $\mathbf{Z}[y][t]$-linearly the Bruhat representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ to that of the algebra $\widetilde{\mathcal{E}}_{n,0}$. In this case, the Dunkl elements commutes with the multiplication by y_i's.

Proposition 3.2 *The list of relations given in Corollary 2.1 describes the complete set of relations among the Dunkl elements $\theta_1^{(n)}, \ldots, \theta_n^{(n)}$ in the algebra $\widetilde{\mathcal{E}}_n[t]$. In other words, the following surjective homomorphism φ between $\mathbf{Z}[t][y_1,\ldots,y_n]$-algebras is an isomorphism:*

$$\begin{array}{rccc}\varphi: & \mathbf{Z}[t][y_1,\ldots,y_n][z_1,\ldots,z_n]/J_n^t & \to & \mathbf{Z}[t][y_1,\ldots,y_n][\theta_1,\ldots,\theta_n] \subset \widetilde{\mathcal{E}}_n[t] \\ & z_i & \mapsto & \theta_i,\end{array}$$

where the ideal J_t is generated by the polynomials

$$e_k(z_1,\ldots,z_n)-e_k(y_1,\ldots,y_n)-\sum_{r\geq 1}(-t)^r\ (2r-1)!!\binom{n-k+2r}{2r}e_{k-2r}(y_1,\ldots,y_n)$$

for $k=1,\ldots,n$.

Proof. For the (single) Schubert polynomial $\mathfrak{S}_w(\theta_1,\ldots,\theta_n)\in\mathcal{E}_n\langle R\rangle[t]$ in the Dunkl elements, we have

$$\mathfrak{S}_w(\theta_1,\ldots,\theta_n)(\underline{\mathrm{id.}})|_{t=0}=\underline{w}+(\text{linear combination of } \underline{v} \text{ with } l(v)<l(w)),$$

so

$$\begin{aligned}&\mathfrak{S}_w(\theta_1,\ldots,\theta_n)(\underline{\mathrm{id.}})\\ &\quad=\ \mathfrak{S}_w(\theta_1,\ldots,\theta_n)(\underline{\mathrm{id.}})|_{t=0}+(\text{linear combination of } \underline{v} \text{ with } l(v)<l(w))\\ &\quad=\ \underline{w}+(\text{linear combination of } \underline{v} \text{ with } l(v)<l(w)).\end{aligned}$$

Hence, $\mathfrak{S}_w(\theta_1,\ldots,\theta_n)$'s are linearly independent in $\mathcal{E}_n\langle R\rangle[t]$ over $R^{\mathbf{S}_n}$.

Let $R_{\mathbf{S}_n}$ be the coinvariant algebra of $\mathbf{S}_n$. Since $R=R^{\mathbf{S}_n}\otimes R_{\mathbf{S}_n}$, the polynomials $\mathfrak{S}_w(z)$, $w\in\mathbf{S}_n$, form a $\mathbf{Z}[z]^{\mathbf{S}_n}$-basis of $\mathbf{Z}[z]$. In particular, any polynomial $f(z_1,\ldots,z_n)\in\mathbf{Z}[z]$ can be expressed as

$$f(z_1,\ldots,z_n)=\sum_{w\in\mathbf{S}_n}\phi_w(z)\mathfrak{S}_w(z),$$

where $\phi_w(z)\in\mathbf{Z}[z]^{\mathbf{S}_n}=\mathbf{Z}[e_1(z),\ldots,e_n(z)]$. Therefore the image of $f(z_1,\ldots,z_n)$ in $\mathbf{Z}[t][y][z_1,\ldots,z_n]/J_n^t$ is a linear combination of $\mathfrak{S}_w(z)$'s over $\mathbf{Z}[t][y]$. Since the elements $\mathfrak{S}_w(\theta_1,\ldots,\theta_n)$ in $\widetilde{\mathcal{E}}_n[t]$ are linearly independent over $\mathbf{Z}[t][y]$, the homomorphism φ is an isomorphism.

Corollary 3.1 *The subalgebra of $\widetilde{\mathcal{E}}_{n,0}$ generated by the Dunkl elements $\theta_1,\ldots,\theta_n$ over $H_T^*(\mathrm{pt})=\mathbf{Z}[y_1,\ldots,y_n]$ is isomorphic to the T-equivariant cohomology ring $H_T^*(Fl_n)$.*

Proof. Let $(0=U_0\subset U_1\subset\cdots\subset U_n)$ be the universal flag over Fl_n. First of all it follows from Corollary 2.1 that the natural map $z_i:=-c_1^T(U_i/U_{i-1})\mapsto\theta_i$, $y_i\mapsto y_i$ defines a surjective homomorphism

$$\pi: H_T^*(Fl_n)\to\mathbf{Z}[y_1,\ldots,y_n][\theta_1,\ldots,\theta_n]\subset\widetilde{\mathcal{E}}_{n,0}.$$

On the other hand, Proposition 3.2 shows that the homomorphism π is injective.

Example 3.1 When $t = 0$, our formula in Theorem 2.1 specializes to the Pieri formula in $H_T^*(Fl_n)$ under the identification in Corollary 3.1. Let $\{\Omega_w\}_{w\in\mathbf{S}_n}$ be the Schubert basis of $H_T^*(Fl_n)$ and $z_i := -c_1^T(U_i/U_{i-1})$. Then our formula in $H_T^*(Fl_n)$ can be written as follows:

$$e_k(z_1,\ldots,z_m)\cdot\Omega_w = \sum_{S\subset\{1,\ldots,m\}}\prod_{i\in S} y_{w(i)}\sum_{(*)}\Omega_{wt_{i_lj_l}\cdots t_{i_1j_1}},$$

where $(*)$ stands for the conditions $|S|+l=k$; $1\le i_a\le m<j_a\le n$ for $1\le a\le l$; $i_1,\ldots,i_l$ are distinct; $j_1\le\cdots\le j_l$; there exists a path $w\to wt_{i_lj_l}\to wt_{i_lj_l}t_{i_{l-1}j_{l-1}}\to\cdots\to wt_{i_lj_l}\cdots t_{i_1j_1}$ in the Bruhat ordering of $\mathbf{S}_n$. For the cyclic permutation $[m,k] := s_{m-k+1}s_{m-k+2}\cdots s_m$, the corresponding double Schubert polynomial is given as follows (see [11, Proposition 2.6.7]):

$$\mathfrak{S}_{[m,k]}(x,y) = \sum_{j=0}^{k} e_{k-j}(x_1,\ldots,x_m)h_j(-y_1,\ldots,-y_{m-k+1}),$$

where h_j is the complete symmetric polynomial of degree j, so the above formula gives the multiplication rule for $\Omega_{[m,k]}$ in $H_T^*(Fl_n)$.

Now we consider an example from [14]. Let $[5,5] = s_1s_2s_3s_4s_5$ be a permutation in $\mathbf{S}_9$. For $w = s_3s_4s_5s_7s_6s_5$ and $u = wt_{26}t_{16}t_{59}$, let us compute the coefficient $p^u_{[5,5],w}$ of Ω_u in the expansion of $\Omega_{[5,5]}\cdot\Omega_w$ by using our formula. It is easy to see that there exists a unique path of length 3 from w to u as follows:

$$w\to wt_{59}\to wt_{59}t_{26}\to u = wt_{59}t_{26}t_{16}.$$

Hence, the action of $\mathfrak{S}_{[m,k]}(\theta,y)$ on $\underline{w}$ under the Bruhat representation at $t=0$ is given by

$$\begin{aligned}
\mathfrak{S}_{[m,k]}(\theta,y)(\underline{w}) &= \sum_{j=0}^{5}(-y_1)^j e_{5-j}(\theta_1,\ldots,\theta_5)\underline{w}\\
&= (\cdots + (x_3x_4 - y_1(x_3+x_4)+y_1^2)[16][26][59]+\cdots)\underline{w}\\
&= \cdots + (y_1-y_4)(y_1-y_6)\underline{u}+\cdots,
\end{aligned}$$

so we get $p^u_{[5,5],w} = (\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\cdots+\alpha_5)$, $\alpha_i = y_i - y_{i+1}$. This coincides with the result computed in [14, Example 4.8].

4 Quantization

Definition 4.1 *The algebra* $\mathcal{E}_n^{\mathbf{q}}$ *is a* $\mathbf{Z}[q_{ij} = q_{ji} | 1 \le i < j \le n]$*-algebra defined by the same generators and relations as in the definition of the algebra* $\mathcal{E}_n$ *except that the relation* (1) *is replaced by*

$$(1)' \quad [i,j]^2 = q_{ij}$$

for $1 \le i < j \le n$*. The algebra* $\mathcal{E}_n^q$ *is defined as a* $\mathbf{Z}[q_1, \ldots, q_{n-1}]$*-algebra obtained from* $\mathcal{E}_n^{\mathbf{q}}$ *by the specialization*

$$q_{ij} = \begin{cases} q_i & \text{if } i = j-1, \\ 0, & \text{if } i < j-1. \end{cases}$$

The extension $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$ *(resp.* $\mathcal{E}_n^q\langle R\rangle[t]$*) of the algebra* $\mathcal{E}_n^{\mathbf{q}}$ *(resp.* $\mathcal{E}_n^q$*) is also defined by the relations* (A)*,* (B) *and* (C)*.*

In the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$, we have an analogous formula to Theorem 2.1. In order to state the formula, we need the quantum elementary symmetric polynomials $e_k^{\mathbf{q}}$ defined by the recursive formula

$$e_k^{\mathbf{q}}(X_i | i \in I \cup \{j\}) = e_k^{\mathbf{q}}(X_i | i \in I) + X_j e_{k-1}^{\mathbf{q}}(X_i | i \in I) + \sum_{a \in I} q_{aj} e_{k-2}^{\mathbf{q}}(X_i | i \in I \setminus \{a\}),$$

$$e_0^{\mathbf{q}}(X_i | i \in I) = 1, \quad e_k^{\mathbf{q}}(\emptyset) = 0, \; k > 0,$$

where I is a subset of $\{1, \ldots, n\}$ and $j \notin I$. The Dunkl elements $\theta_i = \theta_i^{(n)}$ in $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$ are defined by the same formula in the classical case.

Theorem 4.1 For $k \le m \le n$, we have the following formula in the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$:

$$e_k^{\mathbf{q}}(\theta_1^{(n)}, \ldots, \theta_m^{(n)}) =$$

$$\sum_{r \ge 0} (-t)^r \; N(m-k, 2r) \Big\{ \sum_{S, I = \{i_a\}, (j_a)} X_S \cdot [i_1, j_1] \cdots [i_{|I|}, j_{|I|}] \Big\},$$

where the second summation runs over triples $(S, I = \{i_1, \ldots, i_{|I|}\}, (j_a)_{a=1}^{|I|})$ such that $S \subset \{1, \ldots, m\}$; I is a subset of $\{1, \ldots, m\} \setminus S$; $|I| + |S| + 2r = k$; $1 \le i_a \le m < j_a \le n$; $j_1 \le \cdots \le j_{|I|}$.

Proof. We use the same symbols as in the proof of Theorem 2.1. We will show that $E_k(\mathcal{A})$ satisfies the recursive relation

$$E_k(\mathcal{A}\cup\{j\}) = E_k(\mathcal{A}) + E_{k-1}(\mathcal{A})(x_j + \sum_{s\neq j}[j,s]) + \sum_{\nu\in\mathcal{A}} q_{\nu j}E_{k-2}(\mathcal{A}\setminus\{\nu\})$$

in the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$. All the arguments in the proof of Theorem 2.1 work well except that $C_4^r = 0$, so

$$E_k(\mathcal{A}\cup\{j\}) - E_k(\mathcal{A}) - E_{k-1}(\mathcal{A})(x_j + \sum_{s\neq j}[j,s]) = -\sum_{r\geq 0}(-t)^r N(m-k+1,2r)C_4^r.$$

The following cyclic relation ([13, Lemma 5.3]) holds in the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R\rangle[t]$:

$$\begin{aligned}&\sum_{k=1}^m [a,i_k][a,i_{k+1}]\cdots[a,i_m]\cdot[a,i_1]\cdots[a,i_{k-1}][a,i_k]\\ &\quad = \sum_{k=1}^m q_{ak}[i_k,i_{k+1}][i_k,i_{k+2}]\cdots[i_k,i_m][i_k,i_1]\cdots[i_k,i_{k-1}],\end{aligned}$$

where $1 \leq a, i_1, \ldots, i_m \leq n$ are distinct. By using this cyclic relation, we have

$$\begin{aligned}&-\sum_{r\geq 0}(-t)^r N(m-k+1,2r)C_4^r\\ &= -\sum_{r\geq 0}(-t)^r N(m-k+1,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S} \langle\langle I_1|j_1\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle \sum_{s\in I_1}[j,s]\\ &= \sum_{r\geq 0}(-t)^r N(m-k+1,2r)\sum_{S\subset\mathcal{A}} X_S \sum_{\nu\in\mathcal{A}\setminus S}\sum_{(*)} q_{j\nu}\langle\langle I_1|j_1\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle\langle\langle I_{d+1}|\nu\rangle\rangle\\ &= \sum_{\nu\in\mathcal{A}} q_{j\nu}\sum_{r\geq 0}(-t)^r N((m-1)-(k-2),2r)\sum_{S\subset\mathcal{A}\setminus\{\nu\}} X_S \sum_{(*)}\langle\langle I_1|j_1\rangle\rangle\cdots\langle\langle I_{d+1}|j_{d+1}\rangle\rangle\\ &= \sum_{\nu\in\mathcal{A}} q_{j\nu}E_{k-2}(\mathcal{A}\setminus\{\nu\}),\end{aligned}$$

where $(*)$ means the condition $I_1\cdots I_{d+1} \subset_{k-2-2r-|S|} \mathcal{A}\setminus(S\cup\{\nu\})$. This completes the proof.

The Bruhat representation for $\mathcal{E}_n$ is deformed to the quantum Bruhat representation for $\mathcal{E}_n^q$. We define the quantum Bruhat operator σ_{ij}^q, $i < j$,

acting on $\mathbf{Z}[q_1,\ldots,q_{n-1}]\langle \mathbf{S}_n\rangle = \oplus_{w\in\mathbf{S}_n}\mathbf{Z}[q_1,\ldots,q_{n-1}]\cdot \underline{w}$ as follows:

$$\sigma^q_{ij}(\underline{w}) = \begin{cases} q_{ij}\underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) - 2(j-i) + 1, \\ \underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ \underline{0}, & \text{otherwise.} \end{cases}$$

For $f(y) \in \mathbf{Z}[y_1,\ldots,y_n][t]$ and $w \in \mathbf{S}_n$, we define the $\mathbf{Z}[q_1,\ldots,q_{n-1}][t]$-linear operators $\tilde{\sigma}^q_{ij}$ by

$$\tilde{\sigma}^q_{ij}(f(y)\underline{w}) = t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\sigma^q_{ij}(\underline{w}).$$

We can check the well-definedness of the quantum extended Bruhat representation $[ij] \mapsto \tilde{\sigma}^q_{ij}$, $x_k \mapsto \xi_k$ of the algebra $\mathcal{E}^q_n\langle R\rangle[t]$ in the same way as the proof of Proposition 3.1.

Theorem 4.2 *Let $\mathfrak{S}^q_w(x,y)$ be the quantum double Schubert polynomial corresponding to $w \in \mathbf{S}_n$ (see* [3] *and* [8]*). When $t = 0$, we have*

$$\mathfrak{S}^q_w(\theta,y)(\underline{\mathrm{id.}}) = \underline{w}$$

under the quantum extended Bruhat representation.

Proof. This follows from the quantum Monk formula [5] for the quantum Schubert polynomials.

Corollary 4.1 *The quantum double Schubert polynomials $\mathfrak{S}^q_w(x,y)$ are characterized by the conditions:*
(1) $\mathfrak{S}^q_w(x,y)|_{q=0} = \mathfrak{S}_w(x,y)$,
(2) $\mathfrak{S}^q_w(x,y)$ *is a linear combination of polynomials $\mathfrak{S}_v(x,y)$ with $v \le w$ over* $\mathbf{Z}[q_1,\ldots,q_{n-1}]$,
(3) $\mathfrak{S}^q_w(\theta,y)(\underline{\mathrm{id.}}) = \underline{w}$ *under the quantum extended Bruhat representation at* $t = 0$.

Proof. Let $\{P_w(x,y)\}_{w\in\mathbf{S}_n}$ be a family of polynomials which satisfy the above properties (1), (2) and (3) of the quantum Schubert polynomials. We will show that $P_w(x,y) = \mathfrak{S}^q_w(x,y)$ for all $w \in \mathbf{S}_n$. Let $S = 1 + (q_1,\ldots,q_{n-1})$ be a multiplicative system of $\mathbf{Z}[q_1,\ldots,q_{n-1}]$ consisting of polynomials of form $1 + \sum_{i=1}^{n-1} q_i a_i(q)$, $a_i(q) \in \mathbf{Z}[q_1,\ldots,q_{n-1}]$. The properties (1) and (2) imply that

$$\mathfrak{S}^q_w(x,y) = \sum_{v\le w} b^v_w(q)\mathfrak{S}_v(x,y), \quad b^v_w(q) \in \mathbf{Z}[q_1,\ldots,q_{n-1}],\ b^w_w(q) \in S.$$

Then it is easy to see that

$$\mathfrak{S}_w(x,y) = \sum_{v \le w} c_w^v(q)\mathfrak{S}_v^q(x,y), \quad c_w^v(q) \in S^{-1}\mathbf{Z}[q_1, \ldots, q_{n-1}],$$

by induction on the Bruhat ordering. Similarly we also have

$$\mathfrak{S}_w(x,y) = \sum_{v \le w} d_w^v(q)P_v(x,y), \quad d_w^v(q) \in S^{-1}\mathbf{Z}[q_1, \ldots, q_{n-1}].$$

Hence we have

$$\sum_{v \le w} (c_w^v(q)\mathfrak{S}_v^q(x,y) - d_w^v(q)P_v(x,y)) = 0$$

for all $w \in \mathbf{S}_n$. From the property (3), we obtain

$$\sum_{v \le w} (c_w^v(q) - d_w^v(q))\underline{v} = 0$$

for all $w \in \mathbf{S}_n$, so $c_w^v(q) = d_w^v(q)$ for all $w \in \mathbf{S}_n$ and $v \le w$. We can conclude that $P_w(x,y) = \mathfrak{S}_w^q(x,y)$ from these identities again by induction on the Bruhat ordering.

5 Nichols-Woronowicz model

The model of the equivariant cohomology ring $H_T^*(Fl_n)$ in the algebra $\widetilde{\mathcal{E}}_n$ has a natural interpretation in terms of the Nichols-Woronowicz algebra. The Nichols-Woronowicz approach leads us to the uniform construction for arbitrary root systems. For the definition of the Nichols-Woronowicz algebra, see e.g. [1], [2], [12] and [16].

We denote by $\mathcal{B}_W$ the Nichols-Woronowicz algebra associated to the Yetter-Drinfeld module

$$V = \bigoplus_{\alpha \in \Delta} \mathbf{R}[\alpha]/([\alpha] + [-\alpha])$$

over the finite Coxeter group W of the root system Δ. Let $\mathfrak{h}$ be the reflection representation of W and $R = \mathrm{Sym}\mathfrak{h}^*$ the ring of polynomial functions on $\mathfrak{h}$.

Let us consider the extension $\mathcal{B}_W\langle R\rangle[t]$ of the algebra $\mathcal{B}_W$ by the polynomial ring $R[t]$ defined by the commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x,\alpha) \quad \text{for } \; x \in \mathfrak{h}^*.$$

Definition 5.1 *We define the R-algebra $\widetilde{\mathcal{B}}_W$ by*

$$\widetilde{\mathcal{B}}_W = \mathcal{B}_W\langle R\rangle[t] \otimes_{R^W} R.$$

Choose a W-invariant constants $(c_\alpha)_\alpha$. Let us consider a linear map $\mu : \mathfrak{h}^* \to \widetilde{\mathcal{B}}_W$ defined as

$$\mu(x) = x + \sum_{\alpha\in\Delta_+} c_\alpha (x,\alpha)[\alpha]$$

for $x \in \mathfrak{h}^*$.

Proposition 5.1 $[\mu(x), \mu(y)] = 0$, $x, y \in \mathfrak{h}^*$.

Proof. Let $\mu_0(x) := \sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)[\alpha]$. Here we may normalize the length of the roots to have $(\alpha,\alpha) = 1$, $\alpha \in \Delta$. The commutativity $[\mu_0(x), \mu_0(y)] = 0$ has been shown in [2]. The commutativity between $\mu(x)$ and $\mu(y)$ follows from

$$\begin{aligned}
\mu_0(x)y + x\mu_0(y) &= \sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)s_\alpha(y)[\alpha] + t\sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)(y,\alpha) + \sum_{\alpha\in\Delta_+} c_\alpha(y,\alpha)x[\alpha] \\
&= \sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)y[\alpha] - \sum_{\alpha\in\Delta_+} 2c_\alpha(x,\alpha)(y,\alpha)\alpha[\alpha] \\
&\quad + t\sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)(y,\alpha) + \sum_{\alpha\in\Delta_+} c_\alpha(y,\alpha)x[\alpha] \\
&= \sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)y[\alpha] + t\sum_{\alpha\in\Delta_+} c_\alpha(x,\alpha)(y,\alpha) + \sum_{\alpha\in\Delta_+} c_\alpha(y,\alpha)s_\alpha(x)[\alpha] \\
&= y\mu_0(x) + \mu_0(y)x.
\end{aligned}$$

Proposition 5.1 shows that the linear map μ extends to a homomorphism of algebras

$$\mu : R \to \mathcal{B}_W\langle R\rangle[t].$$

Denote by $\widetilde{\mu}$ the composite of the homomorphisms

$$R \otimes_{\mathbf{R}} R \stackrel{\mu\otimes 1}{\to} \mathcal{B}_W\langle R\rangle[t] \otimes_{\mathbf{R}} R \to \widetilde{\mathcal{B}}_W.$$

We will show in Theorem 5.1 below that the image of the algebra homomorphism $\widetilde{\mu}$ at $t=0$ is isomorphic the algebra $R \otimes_{R^W} R$. The proof is based on the correspondence between the twisted derivation D_α and the divided difference operator $\partial_\alpha := (1-s_\alpha)/\alpha$, which acts on the first tensor component of $R \otimes_{R^W} R$ and extends linearly with respect to the second tensor component. We define the operator D_α as the twisted derivation on $\widetilde{\mathcal{B}}_W$ determined by the conditions:
(1): $D_\alpha(x) = 0$, for $x \in R$,
(2): $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(3): $D_\alpha(fg) = D_\alpha(f)g + s_\alpha(f)D_\alpha(g)$.
The operator D_α is linear with respect to R on the second component.

Proposition 5.2

$$\cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha) = R[t] \otimes_{R^W} R.$$

Proof. Since $\mathcal{B}_W\langle R\rangle[t] \cong R[t] \otimes_{\mathbf{R}} \mathcal{B}_W$ as a rignt $\mathcal{B}_W$-module, any element $\omega \in \mathcal{B}_W\langle R\rangle[t]$ can be written as

$$\omega = f_1\varphi_1 + \cdots + f_k\varphi_k,$$

where $f_1, \ldots, f_k \in R[t]$ are linearly independent, and $\varphi_1, \ldots, \varphi_k \in \mathcal{B}_W$. We have

$$D_\alpha(\omega) = s_\alpha(f_1)D_\alpha(\varphi_1) + \cdots + s_\alpha(f_k)D_\alpha(\varphi_k)$$

from the twisted Leibniz rule. If $D_\alpha(\omega) = 0$, we have $D_\alpha(\varphi_1) = \cdots = D_\alpha(\varphi_k) = 0$. Hence, $\omega \in \cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha)$ implies that φ_i belongs to the homogeneous part $\mathcal{B}_W^0 \cong \mathbf{R}$ of degree zero for $i = 1, \ldots, k$. This means $\omega \in R[t]$.

Proposition 5.3

$$D_\alpha(\widetilde{\mu}(x)) = c_\alpha\widetilde{\mu}(\partial_\alpha(x))$$

for $x \in R \otimes_{\mathbf{R}} R$.

Proof. When $x = \beta \otimes 1$, $\beta \in \Delta$, we can check that

$$D_\alpha(\widetilde{\mu}(\beta \otimes 1)) = c_\alpha(\beta, \alpha) = c_\alpha \widetilde{\mu}(\partial_\alpha(\beta)).$$

Hence, we have $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in \mathfrak{h}^* \otimes R$. On the other hand, the both-hands sides satisfy the same twisted Leibniz rule, so it follows that $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in R \otimes R$.

Theorem 5.1 *If $t = 0$ and the constants $(c_\alpha)_\alpha$ are generic, the image of the homomorphism $\widetilde{\mu}$ is isomorphic to the algebra $R \otimes_{R^W} R$. In particular, when W is the Weyl group, it is isomorphic to the T-equivariant cohomology ring $H_T^*(G/B)$ of the corresponding flag variety G/B.*

Proof. If $x \in R^W \otimes_{\mathbf{R}} R$, we have $D_\alpha(\widetilde{\mu}(x)) = 0$ for every $\alpha \in \Delta_+$ from Proposition 5.3. This implies from Proposition 5.2 that $\widetilde{\mu}(x) \in R^W \otimes_{R^W} R$. When $t = 0$, $\widetilde{\mu}(x)$ coincides with the element of R which is obtained by replacing all the symbols $[\alpha]$ by zero in $\widetilde{\mu}(x)$. Hence, the homomorphism $\widetilde{\mu}$ factors through $R \otimes_{R^W} R \to \widetilde{\mathcal{B}}_W$.

Let us take a reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_l}}$ of an element $w \in \mathbf{S}_n$. Then the operator $\partial_w := \partial_{\alpha_{i_1}} \cdots \partial_{\alpha_{i_l}}$ is independent of the choice of reduced decompositions thanks to the Coxeter relation. We also define $D_w := D_{\alpha_{i_1}} \cdots D_{\alpha_{i_l}}$. Define a family $\{X_w\}_{w \in \mathbf{S}_n}$ of polynomials by $X_{w_0} := |W|^{-1} \prod_{\alpha \in \Delta_+} \alpha$ and $X_w := \partial_{w^{-1}w_0} X_{w_0}$. The family $\{X_w\}_{w \in W}$ gives a linear basis of R_W. We can see that

$$CT(\partial_w X_v) = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{otherwise.} \end{cases}$$

where CT stands for the part of degree zero. Since a linear basis $\{X_w\}_{w \in W}$ of the coinvariant algebra of W gives an R^W-basis of R, and $D_w \widetilde{\mu}(X_v \otimes 1) = c_{\alpha_{i_1}} \cdots c_{\alpha_{i_l}} \widetilde{\mu}(\partial_w X_v \otimes 1)$, it is easy to see that $R \otimes_{R^W} R \to \widetilde{\mathcal{B}}_W$ is injective.

Remark 5.1 Our construction is not a straightforward application of the functor $(-) \otimes_{R^W} R$ to the one given in [2] even when $t = 0$. In fact, the defining relations of the algebra $\mathcal{B}_W\langle R\rangle[t]$ involve a nontrivial commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x, \alpha).$$

Moreover, in the formula

$$\mu(x) = x + \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)[\alpha],$$

the first term in the right-hand side does not appear in the non-equivariant case.

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